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Proof of Power Series and Laurent Expansions of Complex Differentiable Functions without Use of Cauchy's Integral Formula or Cauchy's Integral Theorem*

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Let f be a complex function, differentiable throughout an open set S of complex numbers. Then f has derivatives of all orders throughout S and can be expanded in a power series throughout every disk lying in S. These facts have been established by means of Cauchy's integral formula and a natural question has arisen, whether there exists another method of proof, not employing that tool, a formula which is not directly related to the concept of differentiability and whose validity for differentiable functions can be viewed as a fortunate incidence.

In fact, thanks to the work of E. Connell, R. L. Plunkett, P. Porcelli, A. H. Read, and G. T. Whyburn, such a new method of proof [18] based on Topological Analysis has been developed. However, it is a method which deviates from the mainstream procedures of Classical Analysis.

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The following is a very natural approach to the expansion of complex differentiable functions. Suppose $0 \leqslant R' < R'' < \infty$, and let f be a complex function, differentiable in the annulus $R' < |z| < R''$. We would like to prove that f has, throughout that annulus, an expansion

$$f(z) = \sum_{k=1}^{\infty} a_{-k} z^{-k} + \sum_{k=0}^{\infty} a_k z^k. \tag{1}$$

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Choose an r, $R'<r<R''$, and consider the function $f(re^{i\theta})$, $-\infty<\theta<\infty$, of period 2π. This function, being everywhere differentiable, can, by Dini's test [19, p. 52], be expanded everywhere in a Fourier series:

$$f(re^{i\theta})=\lim_{n\to\infty}\sum_{k=-n}^{n} c_k(r)\,e^{ik\theta}, \tag{2}$$

where

$$c_k(r)=(2\pi)^{-1}\int_0^{2\pi} f(re^{i\varphi})\,e^{-ik\varphi}\,d\varphi, \qquad k=0,\ \pm 1,\ \pm 2, ... \tag{3}$$

If $z=re^{i\theta}$, θ real, then from (2),

$$f(z)=\lim_{n\to\infty}\sum_{k=-n}^{n} c_k(r)\,r^{-k}z^k. \tag{4}$$

If one could show, for each fixed k, that, for $R'<r<R''$, $c_k(r)\,r^{-k}$ is a constant a_k, independent of r, then (4) would yield

$$f(z)=\lim_{n\to\infty}\sum_{k=-n}^{n} a_k z^k$$

which is, essentially, (1). Using the polar form of Cauchy–Riemann equations, one can show, in case $f'(z)$ is known to be *continuous* in the annulus $R'<|z|<R''$, by differentiating under the integral sign, that $(d/dr)[c_k(r)\,r^{-k}]=0$ throughout (R', R''), for $k=0$, ± 1, ± 2, ..., and hence, for each such k, $c_k(r)\,r^{-k}$ is indeed a constant in (R', R''). This procedure was carried out by P. R. Beesack [1], and can also be done if $f'(z)$ is only known to be bounded in the annulus and, more generally, if it is merely given that there is a real function $M(\theta)$, summable over $(0, 2\pi)$, such that, for every $r\in(R', R'')$, $\theta\in(0, 2\pi)$, one has $|f'(re^{i\theta})|\leqslant M(\theta)$.

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Our main goal is thus to prove that each $c_k(r)\,r^{-k}$ is independent of r, without making any assumption on f' beyond its existence in the annulus $R'<|z|<R''$. Two features of our proof are: (α) In contrast to the proof of (1) based on Cauchy's integral formula, which is a very "complex analytic" proof, starting a deep schism between much of complex analysis and real analysis, our proof is essentially a "real" one, keeping an intimate relationship between real and complex analysis. Such a close intimacy is of great value in our era of extreme specialization, threatening the unity of

mathematics. Also, our method, unlike the use of topological analysis, belongs in spirit to the mainstream of classical analysis. (β) Our main tool is the Generalized Riemann Integral. It was introduced in the beginning of the century by O. Perron and A. Denjoy, but more recently was given an equivalent definition which is merely a slight variation of the definition of the Riemann integral, making it a very elementary concept. At the same time, it is more powerful than the Lebesgue integral which it includes (together with other integrals) as a special case. There is every reason to make the generalized Riemann integral the standard integral of the working analyst, and textbooks which are essentially doing so are starting to appear: [3] ("The Gauge Integral"), [11] ("The P-Integral"). It is hoped that the present paper will contribute to accelerating this trend.

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To keep our work self-contained, we start by defining the (one-dimensional) generalized Riemann integral and stating some of its fundamental properties. This definition goes back to J. Kurzweil and independently to R. Henstock who has studied this concept extensively. An elementary monograph on the subject is [12] and a rapid survey can be obtained from [2, 9] where the generalized Riemann integral is related to the "Dominated Integral" and the "Simple Integral" introduced and studied by the author and his co-workers [4, 5, 15, 16, 17, 8, 13, 14].

DEFINITION 1. Let $-\infty < a < b < \infty$ and let f be a complex function defined on $[a, b]$. Suppose there is a complex number I having the property: for every $\varepsilon > 0$ there is a positive function $\delta_\varepsilon(x)$ defined on $[a, b]$ such that if

$$a = x_0 < x_1 < \cdots < x_n = b,$$

$$x_{k-1} \leqslant \xi_k \leqslant x_k, \qquad x_k - x_{k-1} < \delta_\varepsilon(\xi_k) \qquad \text{for} \quad k = 1, 2, ..., n,$$

then

$$\left| I - \sum_{k=1}^{n} f(\xi_k)(x_k - x_{k-1}) \right| < \varepsilon.$$

Then this I is unique and is called the generalized Riemann integral of f on $[a, b]$, and f is said to be generalized Riemann integrable on $[a, b]$.

Here are some fundamental properties of this integral.

(I)

$$\int_a^b [\alpha f(x)+\beta g(x)]\,dx=\alpha\int_a^b f(x)\,dx+\beta\int_a^b g(x)\,dx$$

assuming the right-hand side exists, where α, β are any complex constants, and the integrals are generalized Riemann.

(II) If $-\infty<a<b<\infty$, if f is a complex function defined on $[a, b]$, and if $\int_a^b f(x)\,dx$ exists as a (finite) Riemann, Lebesgue, or improper Riemann integral,[1] then it exists also as a generalized Riemann integral, and with the same value.

(III) If f (as a complex function of a real variable) is differentiable at each point of $[a, b]$ $(-\infty<a<b<\infty)$, then

$$\int_a^b f'(x)\,dx=f(b)-f(a) \tag{5}$$

(a generalized Riemann integral). This theorem is false if the integral is taken as Riemann, Lebesgue, or improper Riemann integral.

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Definition 1 has an obvious extension to complex functions on closed n-dimensional intervals, $n=2, 3, \ldots$. For our purposes, we need, however, a variant essentially given in [10], namely,

DEFINITION 2. Let $-\infty<a<b<\infty$, $-\infty<c<d<\infty$, and let f be a complex function defined on the rectangle

$$S=\{(x, y)\colon a\leqslant x\leqslant b,\ c\leqslant y\leqslant d\}$$

in (real) Euclidean 2-space. Suppose there is a complex number I having the property: for every $\varepsilon>0$ there is a positive function $\delta(P)\equiv\delta_\varepsilon(P)$ defined on S such that if a finite set σ of disjoint open subrectangles of S is given, each one, s, containing in its closure, $\bar{s}$, a point $P(s)$ and of the form

$$s=\{(x, y)\colon a_s\leqslant x\leqslant b_s,\ c_s\leqslant y\leqslant d_s\}, \tag{6}$$

where

$$b_s-a_s<\delta(P(s)), \qquad d_s-c_s<\delta(P(s)), \tag{7}$$

$$(d_s-c_s)/(b_s-a_s)=(d-c)/(b-a), \tag{8}$$

[1] The latter, with any finite number of "singular points," i.e., as in [7], Definition 91, p. 323 (for a complex function).

so that

$$\bigcup_{s \in \sigma} \bar{s} = S, \tag{9}$$

then

$$\left| I - \sum_{s \in \sigma} f(P(s))(b_s - a_s)(d_s - c_s) \right| < \varepsilon.$$

Then this I is unique (see below) and we denote it

$$\widehat{\iint}_{\substack{a \leqslant x \leqslant b \\ c \leqslant y \leqslant d}} f(x, y)\, dx\, dy.$$

For the convenience of the reader we supply here a proof of the (known) uniqueness claimed in the last sentence. We first prove the following (known) result: Suppose $\delta(P)$ is a positive function defined on S. Then $(*)$ there is a finite set σ of disjoint open subrectangles of S, each one, s, containing in its closure, $\bar{s}$, a point $P(s)$ and of the form (6), where (7), (8), so that (9). Indeed, assume $(*)$ is false. Using a horizontal and a vertical line bisecting the sides of S, break S into four closed rectangles with interiors mutually disjoint. For at least one of the four, say S_1, $(*)$ with S replaced by S_1, is false. Now break S_1 similarly and arrive at a closed rectangle S_2 for which $(*)$, with S replaced by S_2, is false, and so continue. Let P^* be the point common to all S_k. Let n be a positive integer with

$$(b-a)/2^n < \delta(P^*), \qquad (d-c)/2^n < \delta(P^*).$$

Then $(*)$, with S replaced by S_n, is true, for one can take as σ the singleton consisting of the interior of S_n with which we associate the point P^*. We have thus arrived at a contradiction with the definition of S_n.

If both I_1 and I_2 $(\neq I_1)$ enjoy the property of I in Definition 2, with corresponding functions $\delta_\varepsilon^{(1)}(P)$, $\delta_\varepsilon^{(2)}(P)$, then set

$$\delta(P) \equiv \min(\delta^{(1)}_{|I_1 - I_2|/2}(P), \delta^{(2)}_{|I_1 - I_2|/2}(P)).$$

Using $(*)$, choose σ and points $P(s)$ as in $(*)$. Then

$$\left| I_1 - \sum_{s \in \sigma} f(P(s))(b_s - a_s)(d_s - c_s) \right| < |I_1 - I_2|/2,$$

$$\left| I_2 - \sum_{s \in \sigma} f(P(s))(b_s - a_s)(d_s - c_s) \right| < |I_1 - I_2|/2$$

and hence

$$|I_1 - I_2| < |I_1 - I_2|.$$

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FUNDAMENTAL LEMMA. *Let* $0 < R_1 < R < \infty$ *and let* $f(z)$ *be a complex function, differentiable at each point of the annulus*

$$R_1 \leqslant |z| \leqslant R.$$

Then

$$\iint_{\substack{R_1 \leqslant r \leqslant R \\ 0 \leqslant \varphi \leqslant 2\pi}} f'(re^{i\varphi})\, dr\, d\varphi = \int_0^{2\pi} e^{-i\varphi}[f(Re^{i\varphi}) - f(R_1 e^{i\varphi})]\, d\varphi, \tag{10}$$

$$\widehat{\iint}_{\substack{R_1 \leqslant r \leqslant R \\ 0 \leqslant \varphi \leqslant 2\pi}} f'(re^{i\varphi})\, dr\, d\varphi = \int_{R_1}^{R} r^{-1} \int_0^{2\pi} e^{-i\varphi} f(re^{i\varphi})\, d\varphi\, dr. \tag{11}$$

(*The* (*Riemann*) *integrals on the right in* (10) *and* (11) *clearly exist.*)

We postpone the proof to Sections 8 and 10. We recall here the

POLAR ANALOG OF CAUCHY–RIEMANN EQUATIONS. *Let* f *be a complex function of a complex variable, differentiable at* $z_0 = r_0 e^{i\varphi_0}$(r_0, φ_0 *real*). *Then* (r, φ *considered real variables*) $\partial f(re^{i\varphi})/\partial r$ *and* $\partial f(re^{i\varphi})/\partial \varphi$ *exist at* (r_0, φ_0) *with values*:

$$\partial f(r_0 e^{i\varphi_0})/\partial r = e^{i\varphi_0} f'(z_0),$$

$$\partial f(r_0 e^{i\varphi_0})/\partial \varphi = iz_0 f'(z_0) = ir_0\, \partial f(r_0 e^{i\varphi_0})/\partial r. \tag{12}$$

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We are now in a position to carry out our aim, namely, to re-prove the classical

THEOREM. *Let* $0 \leqslant R' < R'' < \infty$ *and let* $f(z)$ *be a complex function, differentiable in the annulus* $R' < |z| < R''$. *Then there are complex numbers*

$$\ldots, a_{-2}, a_{-1}, a_0, a_1, a_2, \ldots$$

such that, throughout that annulus,

$$f(z) = \sum_{k=0}^{\infty} a_k z^k + \sum_{k=1}^{\infty} a_{-k} z^{-k}. \tag{13}$$

Proof of the Theorem. Let $R' < r < R''$ and set

$$c_k(r) = c_k(r, f) = (2\pi)^{-1} \int_0^{2\pi} f(re^{i\varphi})\, e^{-ik\varphi}\, d\varphi; \qquad k = 0, \pm 1, \pm 2, \ldots. \tag{14}$$

Then, (2), for every real θ,

$$f(re^{i\theta}) = \lim_{n \to \infty} \sum_{k=-n}^{n} c_k(r)\, e^{ik\theta}.$$

Let

$$\gamma_k(r) = \gamma_k(r, f) = c_k(r, f)\, r^{-k}; \qquad k = 0, \pm 1, \pm 2, \ldots, \tag{15}$$

so that, for every real θ,

$$f(re^{i\theta}) = \lim_{n \to \infty} \sum_{k=-n}^{n} \gamma_k(r)(re^{i\theta})^k.$$

We shall prove that, for every integer k,

$$\gamma_k(r) \text{ is constant, say } a_k, \text{ on } (R', R''). \tag{16}$$

It would then follow that, if $R' < |z| < R''$, then

$$f(z) = \lim_{n \to \infty} \sum_{k=-n}^{n} a_k z^k.$$

Also, $\sum_{k=0}^{\infty} a_k z^k$ would converge for $|z| < R''$. Indeed, for such z, choose

$$\max(|z|, R') < r < R''.$$

Then, by (16), (15), and (14), for every integer k,

$$|a_k z^k| = |c_k(r)| \cdot (|z|/r)^k \leqslant [\max\{|f(re^{i\varphi})| : 0 \leqslant \varphi \leqslant 2\pi\}](|z|/r)^k$$

and, hence, $\sum_{k=0}^{\infty} a_k z^k$ converges. Thus, throughout the annulus $R' < |z| < R''$, (13).

Let k be an integer. We shall prove (16). By (15) and (14), for $R' < r < R''$,

$$\gamma_k(r, f) = \gamma_1(r, z^{1-k} f(z)).$$

Hence, it is enough to prove (16) with $k = 1$. Observe that $\gamma_1(r)$ is continuous in (R', R'').

If $R' < r < R''$, then for every real φ,

$$-ie^{-i\varphi} f(re^{i\varphi}) = \partial(e^{-i\varphi} f(re^{i\varphi}))/\partial\varphi - e^{-i\varphi}\,\partial f(re^{i\varphi})/\partial\varphi$$

and hence, by (14), (5), and (12),

$$2\pi c_1(r) = -i\int_0^{2\pi} e^{-i\varphi}[\partial f(re^{i\varphi})/\partial\varphi]\,d\varphi = \int_0^{2\pi} re^{-i\varphi}[\partial f(re^{i\varphi})/\partial r]\,d\varphi, \tag{17}$$

where the last two integrals are generalized Riemann.

Let $R' < R_1 < R < R''$. Then, by (15) and (17),

$$\int_{R_1}^{R} \gamma_1(r)\,dr = (2\pi)^{-1}\int_{R_1}^{R}\int_0^{2\pi} e^{-i\varphi}[\partial f(re^{i\varphi})/\partial r]\,d\varphi\,dr.$$

Suppose we could invert the order of integrations on the right-hand side, the inner one (dr) being generalized Riemann, namely,

$$\begin{aligned}\int_{R_1}^{R} \gamma_1(r)\,dr &= (2\pi)^{-1}\int_0^{2\pi} e^{-i\varphi}[f(Re^{i\varphi}) - f(R_1e^{i\varphi})]\,d\varphi\\ &= R\gamma_1(R) - R_1\gamma_1(R_1).\end{aligned} \tag{18}$$

Then we would obtain throughout (R', R''),

$$\gamma_1(R) = \gamma_1(R) + R\gamma_1'(R),$$

namely, $\gamma_1'(R) = 0$ and, so, γ_1 is constant on (R', R'').

Now (18) can be written, by (15) and (14),

$$\int_{R_1}^{R} r^{-1}\int_0^{2\pi} e^{-i\varphi} f(re^{i\varphi})\,d\varphi\,dr = \int_0^{2\pi} e^{-i\varphi}[f(Re^{i\varphi}) - f(R_1e^{i\varphi})]\,d\varphi$$

which is true, by the Fundamental Lemma.

COROLLARY 1. *Let $0 < R < \infty$ and let $f(z)$ be a complex function, differentiable in the disk $|z| < R$. Then there are complex numbers $a_0, a_1, \ldots$ such that, throughout that disk,*

$$f(z) = \sum_{k=0}^{\infty} a_k z^k. \tag{19}$$

Proof. Let $0 < r \leqslant R/2$. By the proof of the preceding Theorem, with $R' = 0$, $R'' = R$, we have (13) whenever $0 < |z| < R$, where

$$a_k = (2\pi r^k)^{-1}\int_0^{2\pi} f(re^{i\varphi})\,e^{-ik\varphi}\,d\varphi, \qquad k = 0, \pm 1, \pm 2, \ldots.$$

Let k be a positive integer. Then

$$|a_{-k}| = \left|(2\pi)^{-1} r^k \int_0^{2\pi} f(re^{i\varphi})\, e^{ik\varphi}\, d\varphi\right| \leqslant r^k \max\{|f(z)|: |z| \leqslant R/2\}.$$

Letting $r \to 0^+$, we conclude $a_{-k} = 0$. Hence (19) whenever $0 < |z| < R$. By continuity of f and of the power series at 0, we have (19) also for $z = 0$.

COROLLARY 2. *Let f be a complex function and c a complex number.* (a) *Let $0 \leqslant R' < R'' < \infty$ and let f be differentiable in the annulus $R' < |z - c| < R''$. Then there are complex numbers $\ldots, a_{-2}, a_{-1}, a_0, a_1, a_2, \ldots$ such that, throughout that annulus,*

$$f(z) = \sum_{k=0}^{\infty} a_k (z-c)^k + \sum_{k=1}^{\infty} a_{-k} (z-c)^{-k}.$$

(b) *Let $0 < R < \infty$ and let f be differentiable in the disk $|z - c| < R$. Then there are complex numbers $a_0, a_1, \ldots$ such that, throughout that disk,*

$$f(z) = \sum_{k=0}^{\infty} a_k (z-c)^k.$$

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Proof of (10). We may even assume $0 \leqslant R_1 < R < \infty$. Let $\varepsilon > 0$. We show: there is a positive function $\delta_\varepsilon(P)$ defined on

$$S = \{(r, \varphi): R_1 \leqslant r \leqslant R, 0 \leqslant \varphi \leqslant 2\pi\}$$

such that if a finite set σ of disjoint open subrectangles of S is given, each one, s, containing in its closure, $\bar{s}$, a point $P(s) = (r(s), \varphi(s))$ and of the form

$$s = \{(r, \varphi): a_s \leqslant r \leqslant b_s, c_s \leqslant \varphi \leqslant d_s\}$$

where

$$b_s - a_s < \delta_\varepsilon(P(s)), \qquad d_s - c_s < \delta_\varepsilon(P(s)), \tag{20}$$

$$(d_s - c_s)/(b_s - a_s) = 2\pi/(R - R_1), \tag{21}$$

so that

$$\bigcup_{s \in \sigma} \bar{s} = S,$$

then

$$\left|\int_0^{2\pi} e^{-i\varphi}[f(Re^{i\varphi})-f(R_1e^{i\varphi})]\,d\varphi-\sum_{s\in\sigma} f'(r(s)\,e^{i\varphi(s)})(b_s-a_s)(d_s-c_s)\right|<\varepsilon. \tag{22}$$

Set

$$\varepsilon_1=\varepsilon/\{8\pi[(2\pi+1)\,R-R_1]\}. \tag{23}$$

Given $P=(r,\varphi)\in S$, let

$$\delta_\varepsilon(P)=\delta_\varepsilon(r,\varphi)=\min(\eta,\eta_{\varepsilon_1}(r,\varphi)/(1+R)), \tag{24}$$

where $\eta>0$ is such that

$$\left|\int_0^{2\pi} e^{-i\varphi}[f(Re^{i\varphi})-f(R_1e^{i\varphi})]\,d\varphi - \sum_{j=1}^{n} e^{-i\varphi_j}[f(Re^{i\varphi_j})-f(R_1e^{i\varphi_j})](\varphi_j-\varphi_{j-1})\right|<\varepsilon/2 \tag{25}$$

whenever $0=\varphi_0<\varphi_1<\cdots<\varphi_n=2\pi$,

$\max\{\varphi_j-\varphi_{j-1}: 1\leqslant j\leqslant n\}<\eta$

and, for every $\zeta>0$, $\eta_\zeta(r,\varphi)>0$ is such that

$$\begin{gathered}|[f(z)-f(re^{i\varphi})](z-re^{i\varphi})^{-1}-f'(re^{i\varphi})|<\zeta\\ \text{whenever } 0<|z-re^{i\varphi}|<\eta_\zeta(r,\varphi).\end{gathered} \tag{26}$$

With this definition of $\delta_\varepsilon(P)$, let σ and points $P(s)$ be as above in this proof. We prove (22).

Let the horizontal sides of the elements of σ, extended, be the lines

$$\varphi=\varphi_j, \qquad j=0,1,2,\ldots,n,$$

where

$$0=\varphi_0<\varphi_1<\cdots<\varphi_n=2\pi.$$

Let $1\leqslant j\leqslant n$ and consider the rectangle

$$\rho_j=\{(r,\varphi): R_1<r<R,\ \varphi_{j-1}<\varphi<\varphi_j\}.$$

Let its vertical partitions, inherited from σ, be segments of the lines

$$r=r_0^{(j)}, \qquad r=r_1^{(j)},\ \ldots,\ r=r_{m_j}^{(j)},$$

where

$$R_1 = r_0^{(j)} < r_1^{(j)} < \cdots < r_{m_j}^{(j)} = R.$$

Let $1 \leqslant k \leqslant m_j$. There is a unique $s = s_{jk} \in \sigma$ of the form

$$\{(r, \varphi): r_{k-1}^{(j)} < r < r_k^{(j)}, \varphi' < \varphi < \varphi''\}$$

intersecting ρ_j, and, clearly, by (20), (24),

$$\varphi_j - \varphi_{j-1} \leqslant \varphi'' - \varphi' < \eta. \tag{27}$$

Consider the corresponding point $P(s) = (r(s), \varphi(s))$. By (20), (24), for $p = k-1, k$, we have

$$\begin{aligned} |r_p^{(j)} e^{i\varphi_j} - r(s)\, e^{i\varphi(s)}| &= |(r_p^{(j)} - r(s))\, e^{i\varphi_j} + r(s)(e^{i\varphi_j} - e^{i\varphi(s)})| \\ &\leqslant |r_p^{(j)} - r(s)| + r(s)|\, \varphi_j - \varphi(s)| \\ &\leqslant r_k^{(j)} - r_{k-1}^{(j)} + R(\varphi'' - \varphi') \\ &< (1+R)\, \delta_\varepsilon(P(s)) \leqslant \eta_{\varepsilon_1}(r(s), \varphi(s)) \end{aligned}$$

and hence, by (26), (21), and (23), we have

$$\begin{aligned} &|f(r_p^{(j)} e^{i\varphi_j}) - f(r(s)\, e^{i\varphi(s)}) - f'(r(s)\, e^{i\varphi(s)})[r_p^{(j)} e^{i\varphi_j} - r(s)\, e^{i\varphi(s)}]| \\ &\quad \leqslant \varepsilon_1 |r_p^{(j)} e^{i\varphi_j} - r(s)\, e^{i\varphi(s)}| \leqslant \varepsilon_1 [r_k^{(j)} - r_{k-1}^{(j)} + R(\varphi'' - \varphi')] \\ &\quad = \varepsilon_1 [1 + R(\varphi'' - \varphi')(r_k^{(j)} - r_{k-1}^{(j)})^{-1}](r_k^{(j)} - r_{k-1}^{(j)}) \\ &\quad = \varepsilon_1 [1 + 2\pi R(R - R_1)^{-1}](r_k^{(j)} - r_{k-1}^{(j)}) \\ &\quad = \varepsilon (r_k^{(j)} - r_{k-1}^{(j)})/[8\pi(R - R_1)]. \end{aligned}$$

Therefore

$$\begin{aligned} &|f(r_k^{(j)} e^{i\varphi_j}) - f(r_{k-1}^{(j)} e^{i\varphi_j}) - f'(r(s)\, e^{i\varphi(s)})(r_k^{(j)} - r_{k-1}^{(j)})\, e^{i\varphi_j}| \\ &\quad \leqslant \varepsilon (r_k^{(j)} - r_{k-1}^{(j)})/[4\pi(R - R_1)]. \end{aligned}$$

Hence

$$\left| e^{-i\varphi_j}[f(Re^{i\varphi_j}) - f(R_1 e^{i\varphi_j})] - \sum_{k=1}^{m_j} f'(r(s_{jk})\, e^{i\varphi(s_{jk})})(r_k^{(j)} - r_{k-1}^{(j)}) \right| \leqslant \varepsilon/(4\pi),$$

$$\begin{aligned} &\left| \sum_{j=1}^{n} e^{-i\varphi_j}[f(Re^{i\varphi_j}) - f(R_1 e^{i\varphi_j})](\varphi_j - \varphi_{j-1}) \right. \\ &\qquad \left. - \sum_{j=1}^{n} \sum_{k=1}^{m_j} f'(r(s_{jk})\, e^{i\varphi(s_{jk})})(r_k^{(j)} - r_{k-1}^{(j)})(\varphi_j - \varphi_{j-1}) \right| \\ &\quad \leqslant \sum_{j=1}^{n} \varepsilon (4\pi)^{-1} (\varphi_j - \varphi_{j-1}) = \varepsilon/2. \end{aligned}$$

By (25) and (27),

$$\left|\int_0^{2\pi} e^{-i\varphi}[f(Re^{i\varphi})-f(R_1e^{i\varphi})]\,d\varphi - \sum_{j=1}^{n} e^{-i\varphi_j}[f(Re^{i\varphi_j})-f(R_1e^{i\varphi_j})](\varphi_j-\varphi_{j-1})\right| < \varepsilon/2.$$

Therefore

$$\left|\int_0^{2\pi} e^{-i\varphi}[f(Re^{i\varphi})-f(R_1e^{i\varphi})]\,d\varphi - \sum_{s\in\sigma} f'(r(s)\,e^{i\varphi(s)})(b_s-a_s)(d_s-c_s)\right|$$

$$=\left|\int_0^{2\pi} e^{-i\varphi}[f(Re^{i\varphi})-f(R_1e^{i\varphi})]\,d\varphi - \sum_{j=1}^{n}\sum_{k=1}^{m_j} f'(r(s_{jk})\,e^{i\varphi(s_{jk})})(r_k^{(j)}-r_{k-1}^{(j)})(\varphi_j-\varphi_{j-1})\right| < \varepsilon.$$

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To prove (11), we need two simple lemmas.

LEMMA 1. *Let C be a compact (nonempty) set of complex numbers and let g be a complex function, continuous on C (in the sense that if $z, z_1, z_2, \ldots \in C$ and $z_n \to z$, then $f(z_n) \to f(z)$). Let $\eta > 0$. There is $\rho > 0$ such that if*

$$0=\varphi_0<\varphi_1<\cdots<\varphi_n=2\pi,$$
$$\varphi_k-\varphi_{k-1}<\rho, \qquad \varphi_{k-1}\leqslant\alpha_k\leqslant\varphi_k; \qquad k=1,2,\ldots,n, \tag{28}$$

then for every $r>0$ such that the circle $|z|=r$ lies in C, we have

$$m \stackrel{\text{def}}{=} \left|\int_0^{2\pi} g(re^{i\varphi})\,d\varphi - \sum_{k=1}^{n} g(re^{i\alpha_k})(\varphi_k-\varphi_{k-1})\right| < \eta.$$

Proof of Lemma 1. Let $\Delta>0$ be such that

$$|g(z_2)-g(z_1)|<\eta/(4\pi) \qquad \text{whenever } z_1, z_2\in C,\ |z_2-z_1|<\Delta$$

and let

$$\rho=\Delta/[1+\max\{|z|: z\in C\}].$$

Suppose (28) and that the circle $|z| = r$, $r > 0$, lies in C. Then

$$m = \Bigg| \sum_{k=1}^{n} \int_{\varphi_{k-1}}^{\varphi_k} \operatorname{Re} g(re^{i\varphi})\, d\varphi - \{\operatorname{Re} g(re^{i\alpha_k})\}(\varphi_k - \varphi_{k-1})$$

$$+ i \left[\int_{\varphi_{k-1}}^{\varphi_k} \operatorname{Im} g(re^{i\varphi})\, d\varphi - \{\operatorname{Im} g(re^{i\alpha_k})\}(\varphi_k - \varphi_{k-1}) \right] \Bigg|$$

$$= \left| \sum_{k=1}^{n} \{\operatorname{Re}[g(re^{i\beta_k}) - g(re^{i\alpha_k})] + i \operatorname{Im}[g(re^{i\gamma_k}) - g(re^{i\alpha_k})]\}(\varphi_k - \varphi_{k-1}) \right|$$

$$\leqslant \sum_{k=1}^{n} \{|g(re^{i\beta_k}) - g(re^{i\alpha_k})| + |g(re^{i\gamma_k}) - g(re^{i\alpha_k})|\}(\varphi_k - \varphi_{k-1}),$$

where, for $k = 1, 2, ..., n$,

$$\varphi_{k-1} \leqslant \beta_k \leqslant \varphi_k, \qquad \varphi_{k-1} \leqslant \gamma_k \leqslant \varphi_k.$$

For $k = 1, 2, ...$,

$$|re^{i\beta_k} - re^{i\alpha_k}| \leqslant r|\beta_k - \alpha_k| \leqslant r(\varphi_k - \varphi_{k-1}) < r\rho < \Delta$$

and, similarly,

$$|re^{i\gamma_k} - re^{i\alpha_k}| < \Delta.$$

Hence

$$m < \sum_{k=1}^{n} \eta(2\pi)^{-1}(\varphi_k - \varphi_{k-1}) = \eta.$$

LEMMA 2. *There is λ, $0 < \lambda \leqslant \pi$, and a complex function $B(z)$ such that, whenever u, v are complex numbers satisfying $0 < |u| < \lambda$, $0 < |v| < \lambda$, we have*

$$[u/(e^{-iu} - 1)] + [v/(e^{iv} - 1)] = (-1/2)(u + v) + u^2 B(u) - v^2 B(v),$$

where $|B(u)| < 1$, $|B(v)| < 1$.

Proof. It is an elementary fact [6, pp. 117–118] not requiring for its proof any knowledge of differentiable functions of a complex variable, that for some $\tau > 0$ and some numbers B_0, B_1, B_2, ... (Bernoulli's numbers),

$$z/(e^z - 1) = \sum_{n=0}^{\infty} (B_n/n!)\, z^n \qquad \text{whenever } 0 < |z| < \tau,$$

where $B_0 = 1$, $B_1 = -1/2$, $B_2 = 1/16$, and $B_{2n+1} = 0$ for $n = 1, 2, \ldots$. Thus, if $0 < |u| < \tau$, $0 < |v| < \tau$, then

$$[u/(e^{-iu} - 1)] + [v/(e^{iv} - 1)] = i\{[-iu/(e^{-iu} - 1)] - [iv/(e^{iv} - 1)]\}$$
$$= (-1/2)(u + v) + u^2 B(u) - v^2 B(v),$$

where

$$B(z) \equiv \sum_{n=0}^{\infty} (-1)^{n+1} i[B_{2n+2}/(2n+2)!]\, z^{2n}, \qquad |B(0)| = 1/12 < 1$$

which implies the lemma.

10

Proof of (11). Let $\varepsilon > 0$. We repeat the third sentence of Section 8 ("We show: ...") with (22) replaced by

$$\left| \int_{R_1}^{R} r^{-1} \int_0^{2\pi} e^{-i\varphi} f(re^{i\varphi})\, d\varphi\, dr - \sum_{s \in \sigma} f'(r(s)\, e^{i\varphi(s)})(b_s - a_s)(d_s - c_s) \right| < \varepsilon. \quad (29)$$

Set

$$M = \max\{|f(z)|: R_1 \leqslant |z| \leqslant R\}, \qquad \varepsilon_2 = \varepsilon R_1/[16\pi(M+1)(R - R_1)],$$
$$\varepsilon_3 = R_1 \varepsilon/\{8\pi(R - R_1)[(2\pi + 1)\, R - R_1]\}; \quad (30)$$

λ is as in Lemma 2, $\mu > 0$ is such that

$$\left| \int_{R_1}^{R} r^{-1} \int_0^{2\pi} e^{-i\varphi} f(re^{i\varphi})\, d\varphi\, dr - \sum_{j=1}^{n} r_j^{-1} \int_0^{2\pi} e^{-i\varphi} f(r_j e^{i\varphi})\, d\varphi (r_j - r_{j-1}) \right| < \varepsilon/2 \quad (31)$$

whenever $R_1 = r_0 < r_1 \cdots < r_n = R$,

$\max\{r_j - r_{j-1}: 1 \leqslant j \leqslant n\} < \mu$;

$\rho > 0$ is such that (cf. Lemma 1) if (28), then for every $r \in [R_1, R]$,

$$\left| \int_0^{2\pi} f(re^{i\varphi})(re^{i\varphi})^{-1}\, d\varphi - \sum_{k=1}^{n} f(re^{i\alpha_k})(re^{i\alpha_k})^{-1}(\varphi_k - \varphi_{k-1}) \right| < \varepsilon/[8(R - R_1)]; \quad (32)$$

and $\eta_\zeta(r, \varphi)$ is as in (26).

Given $P=(r,\varphi)\in S$, let

$$\delta_\varepsilon(P)=\delta_\varepsilon(r,\varphi)=\min(\varepsilon_2,\lambda,\mu,\rho,\eta_{\varepsilon_3}(r,\varphi)/(R+1)). \tag{33}$$

With this definition, let σ and points $P(s)$ be as in the third sentence of Section 8. We prove (29).

Let the vertical sides of the elements of σ, extended, be the lines

$$r=r_j, \qquad j=0,1,...,n,$$

where

$$R_1=r_0<r_1<\cdots<r_n=R.$$

Let $1\leqslant j\leqslant n$ and consider the rectangle

$$\sigma_j=\{(r,\varphi): r_{j-1}<r<r_j, 0<\varphi<2\pi\}.$$

Let its horizontal partitions, inherited from σ, be segments of the lines

$$\varphi=\varphi_0^{(j)}, \qquad \varphi=\varphi_1^{(j)},...,\varphi=\varphi_{\mu_j}^{(j)},$$

where

$$0=\varphi_0^{(j)}<\varphi_1^{(j)}<\cdots<\varphi_{\mu_j}^{(j)}=2\pi.$$

Let $1\leqslant k\leqslant\mu_j$. There is a unique $s=s_{jk}\in\sigma$ of the form

$$\{(r,\varphi): r'<r<r'', \varphi_{k-1}^{(j)}<\varphi<\varphi_k^{(j)}\}$$

intersecting σ_j and, clearly, by (20), (33),

$$r_j-r_{j-1}\leqslant r''-r'<\mu, \qquad \varphi_k^{(j)}-\varphi_{k-1}^{(j)}<\lambda\leqslant\pi. \tag{34}$$

Consider the corresponding point $P(s)=(r(s),\varphi(s))$. By (20), (33), for $p=k-1,k$, we have

$$\begin{aligned}|r_je^{i\varphi_p^{(j)}}-r(s)\,e^{i\varphi(s)}|&=|r_j(e^{i\varphi_p^{(j)}}-e^{i\varphi(s)})+(r_j-r(s))\,e^{i\varphi(s)}|\\&\leqslant R|\varphi_p^{(j)}-\varphi(s)|+r''-r'\leqslant R(\varphi_k^{(j)}-\varphi_{k-1}^{(j)})+r''-r'\\&<(R+1)\,\delta_\varepsilon(P(s))\leqslant\eta_{\varepsilon_3}(r(s),\varphi(s))\end{aligned}$$

and hence, by (26), (21), and (30), we get

$$\begin{aligned}
&|f(r_je^{i\varphi_p^{(j)}})-f(r(s)\,e^{i\varphi(s)})-f'((r(s)\,e^{i\varphi(s)})[r_je^{i\varphi_p^{(j)}}-r(s)\,e^{i\varphi(s)}]|\\
&\quad\leqslant \varepsilon_3|r_je^{i\varphi_p^{(j)}}-r(s)\,e^{i\varphi(s)}|\leqslant \varepsilon_3[R(\varphi_k^{(j)}-\varphi_{k-1}^{(j)})+r''-r']\\
&\quad=\varepsilon_3[R+(r''-r')(\varphi_k^{(j)}-\varphi_{k-1}^{(j)})^{-1}](\varphi_k^{(j)}-\varphi_{k-1}^{(j)})\\
&\quad=\varepsilon_3[R+(R-R_1)(2\pi)^{-1}](\varphi_k^{(j)}-\varphi_{k-1}^{(j)})\\
&\quad=R_1\,\varepsilon[16\pi^2(R-R_1)]^{-1}(\varphi_k^{(j)}-\varphi_{k-1}^{(j)}).
\end{aligned}$$

Therefore

$$\begin{aligned}
&|f(r_je^{i\varphi_k^{(j)}})-f(r_je^{i\varphi_{k-1}^{(j)}})-r_jf'(r(s)\,e^{i\varphi(s)})(e^{i\varphi_k^{(j)}}-e^{i\varphi_{k-1}^{(j)}})|\\
&\quad\leqslant R_1\,\varepsilon[8\pi^2(R-R_1)]^{-1}(\varphi_k^{(j)}-\varphi_{k-1}^{(j)}).
\end{aligned}$$

For $k=1, 2, ..., \mu_j$, set

$$\psi_k^{(j)}=(\varphi_k^{(j)}-\varphi_{k-1}^{(j)})/[e^{i\varphi_k^{(j)}}-e^{i\varphi_{k-1}^{(j)}}]$$

so that

$$\begin{aligned}
&|r_j^{-1}\psi_k^{(j)}[f(r_je^{i\varphi_k^{(j)}})-f(r_je^{i\varphi_{k-1}^{(j)}})]-f'(r(s_{jk})\,e^{i\varphi(s_{jk})}\,(\varphi_k^{(j)}-\varphi_{k-1}^{(j)})|\\
&\quad<\varepsilon[16\pi(R-R_1)]^{-1}(\varphi_k^{(j)}-\varphi_{k-1}^{(j)})
\end{aligned}$$

by (34), as

$$|(y-x)/(e^{iy}-e^{ix})|=[(y-x)/2]/\sin[(y-x)/2]<\frac{\pi}{2}$$

$$\text{if}\quad 0<y-x<\pi;\ x,\ y\ \text{real}.$$

Thus, setting $\psi_0^{(j)}=\psi_{\mu_j}^{(j)}$, we have:

$$\begin{aligned}
&\left|r_j^{-1}\sum_{k=0}^{\mu_j-1}(\psi_k^{(j)}-\psi_{k+1}^{(j)})\,f(r_je^{i\varphi_k^{(j)}})-\sum_{k=1}^{\mu_j}f'(r(s_{jk})\,e^{i\varphi(s_{jk})})(\varphi_k^{(j)}-\varphi_{k-1}^{(j)})\right|\\
&\quad=\left|\sum_{k=1}^{\mu_j}r_j^{-1}\psi_k^{(j)}[f(r_je^{i\varphi_k^{(j)}})-f(r_je^{i\varphi_{k-1}^{(j)}})]-f'(r(s_{jk})\,e^{i\varphi(s_{jk})})(\varphi_k^{(j)}-\varphi_{k-1}^{(j)})\right|\\
&\quad<\varepsilon[8(R-R_1)]^{-1}.
\end{aligned}$$

For $k=0, 1, ..., \mu_j-1$ (with $\varphi_{-1}^{(j)}=\varphi_{\mu_j-1}^{(j)}-2\pi$), by Lemma 2 and (34),

$$\begin{aligned}
\psi_k^{(j)}-\psi_{k+1}^{(j)}&=-e^{-i\varphi_k^{(j)}}\{[(\varphi_k^{(j)}-\varphi_{k-1}^{(j)})/(e^{-i(\varphi_k^{(j)}-\varphi_{k-1}^{(j)})}-1)]\\
&\qquad+[(\varphi_{k+1}^{(j)}-\varphi_k^{(j)})/(e^{i(\varphi_{k+1}^{(j)}-\varphi_k^{(j)})}-1)]\}\\
&=-e^{-i\varphi_k^{(j)}}[(-1/2)(\varphi_{k+1}^{(j)}-\varphi_{k-1}^{(j)})\\
&\qquad+(\varphi_k^{(j)}-\varphi_{k-1}^{(j)})^2\beta_1-(\varphi_{k+1}^{(j)}-\varphi_k^{(j)})^2\beta_2],
\end{aligned}$$

where $|\beta_1| < 1$, $|\beta_2| < 1$. Hence, using (30) and (33), we obtain

$$\left|(2r_j)^{-1} \sum_{k=1}^{\mu_j} e^{-i\varphi_{k-1}^{(j)}} f(r_j e^{i\varphi_{k-1}^{(j)}})(\varphi_k^{(j)} - \varphi_{k-1}^{(j)})\right.$$

$$+ (2r_j)^{-1} \sum_{k=1}^{\mu_j} e^{-i\varphi_k^{(j)}} f(r_j e^{i\varphi_k^{(j)}})(\varphi_k^{(j)} - \varphi_{k-1}^{(j)})$$

$$\left. - \sum_{k=1}^{\mu_j} f'(r(s_{jk})\, e^{i\varphi(s_{jk})})(\varphi_k^{(j)} - \varphi_{k-1}^{(j)})\right|$$

$$= \left|(2r_j)^{-1} \sum_{k=0}^{\mu_j - 1} e^{-i\varphi_k^{(j)}} f(r_j e^{i\varphi_k^{(j)}})(\varphi_{k+1}^{(j)} - \varphi_{k-1}^{(j)})\right.$$

$$\left. - \sum_{k=1}^{\mu_j} f'(r(s_{jk})\, e^{i\varphi(s_{jk})})(\varphi_k^{(j)} - \varphi_{k-1}^{(j)})\right|$$

$$= \left|r_j^{-1} \sum_{k=0}^{\mu_j - 1} (\psi_k^{(j)} - \psi_{k+1}^{(j)})\, f(r_j e^{i\varphi_k^{(j)}})\right.$$

$$- \sum_{k=1}^{\mu_j} f'(r(s_{jk})\, e^{i\varphi(s_{jk})})(\varphi_k^{(j)} - \varphi_{k-1}^{(j)})$$

$$\left. + r_j^{-1} \sum_{k=0}^{\mu_j - 1} e^{-i\varphi_k^{(j)}} f(r_j e^{i\varphi_k^{(j)}})[\beta_1(\varphi_k^{(j)} - \varphi_{k-1}^{(j)})^2 - \beta_2(\varphi_{k+1}^{(j)} - \varphi_k^{(j)})^2]\right|$$

$$< \varepsilon[8(R - R_1)]^{-1} + R_1^{-1} \sum_{k=0}^{\mu_j - 1} M\varepsilon_2[(\varphi_k^{(j)} - \varphi_{k-1}^{(j)}) + (\varphi_{k+1}^{(j)} - \varphi_k^{(j)})]$$

$$= \varepsilon[8(R - R_1)]^{-1} + 4\pi R_1^{-1} M\varepsilon_2 < 3\varepsilon[8(R - R_1)]^{-1}.$$

As $\varphi_k^{(j)} - \varphi_{k-1}^{(j)} < \rho$, $k = 1, 2, ..., n_j$, we have by (32):

$$\left|r_j^{-1} \int_0^{2\pi} e^{-i\varphi} f(r_j e^{i\varphi})\, d\varphi - (2r_j)^{-1} \sum_{k=1}^{\mu_j} e^{-i\varphi_{k-1}^{(j)}} f(r_j e^{i\varphi_{k-1}^{(j)}})(\varphi_k^{(j)} - \varphi_{k-1}^{(j)})\right.$$

$$\left. - (2r_j)^{-1} \sum_{k=1}^{\mu_j} e^{-i\varphi_k^{(j)}} f(r_j e^{i\varphi_k^{(j)}})(\varphi_k^{(j)} - \varphi_{k-1}^{(j)})\right| < \varepsilon[8(R - R_1)]^{-1}.$$

Thus

$$\left|r_j^{-1} \int_0^{2\pi} e^{-i\varphi} f(r_j e^{i\varphi})\, d\varphi - \sum_{k=1}^{\mu_j} f'(r(s_{jk})\, e^{i\varphi(s_{jk})})(\varphi_k^{(j)} - \varphi_{k-1}^{(j)})\right|$$

$$< \varepsilon[2(R - R_1)]^{-1}.$$

Therefore

$$\left| \sum_{j=1}^{n} r_j^{-1} \int_0^{2\pi} e^{-i\varphi} f(r_j e^{i\varphi})\, d\varphi (r_j - r_{j-1}) \right.$$
$$\left. - \sum_{j=1}^{n} \sum_{k=1}^{\mu_j} f'(r(s_{jk})\, e^{i\varphi(s_{jk})})(\varphi_k^{(j)} - \varphi_{k-1}^{(j)})(r_j - r_{j-1}) \right| < \varepsilon/2.$$

By (34) and (31),

$$\left| \int_{R_1}^{R} r^{-1} \int_0^{2\pi} e^{-i\varphi} f(re^{i\varphi})\, d\varphi\, dr - \sum_{j=1}^{n} r_j^{-1} \int_0^{2\pi} e^{-i\varphi} f(r_j e^{i\varphi})\, d\varphi (r_j - r_{j-1}) \right| < \varepsilon/2$$

and consequently

$$\left| \int_{R_1}^{R} r^{-1} \int_0^{2\pi} e^{-i\varphi} f(re^{i\varphi})\, d\varphi\, dr - \sum_{s \in \sigma} f'(r(s)\, e^{i\varphi(s)})(b_s - a_s)(d_s - c_s) \right|$$
$$= \left| \int_{R_1}^{R} r^{-1} \int_0^{2\pi} e^{-i\varphi} f(re^{i\varphi})\, d\varphi\, dr \right.$$
$$\left. - \sum_{j=1}^{n} \sum_{k=1}^{\mu_j} f'(r(s_{jk})\, e^{i\varphi(s_{jk})})(\varphi_k^{(j)} - \varphi_{k-1}^{(j)})(r_j - r_{j-1}) \right| < \varepsilon.$$

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